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Numerical Scheme for Solution of an Approximation of Saint-Venant Equation

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Abstract

In the work [4] the Saint-Venant's one-dimensional differential model and kinematic approximation describing the free movement of water in an open channel is under consideration. This model is widely used for the calculation of the flow of rivers in the various fields of engineering. Quality of engineering projects depends on the accuracy and speed of the mathematical calculations. This paper presents a promising method for the numerical solution of one approximation of the Saint-Venant's model.

Keywords: Open channels, numerical method, Seint-Venant

Introduction

In [4] we derived kinematic approximation of the Saint-Venant's model described by the equation:

$$\frac{\partial(wh)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\sqrt{|S_0|}}{n}wh^{5/3}\right)\mathrm{sgn}(S_0)=q \tag{1}$$

also one particular solution was founded:

$x_{left}=0$, $x_{right}\to\infty$, w, S_0, n – constants, $q\equiv 0$, $h(0,t)=h_1$ for $t>0$, $h(x,0)=h_2$ for $x>0$.

Using these results we construct a numerical scheme to solve the kinematic equation (1).

Physical model

We represent the linear portion of the river in the form of a chain of open water tanks with a free surface; throughout experiencing the same atmospheric pressure (see Fig. 1). Each reservoir i has a finite length Δx_i, it is made of an impermeable material, it has smooth relatively high vertical walls and

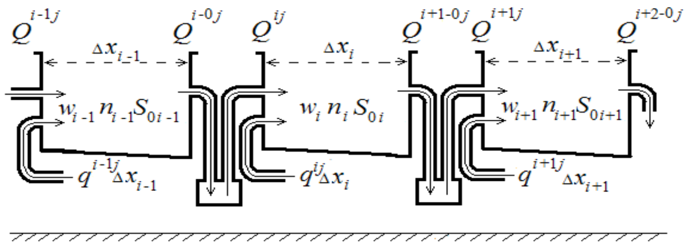

Fig. 1 Chain of tanks

rectangular (w_i), everywhere with equal bottom roughness (n_i) and a small positive slope (S_{0i}). Each tank has two inlet ports in the left side - through which water is pumped into the tank, and one outlet opening in the wall of the right - through which water is pumped from the tank. The outlet of any reservoir i, except the last, through the storage tank is connected to one of the tank inputs $i+1$ (in Fig. 6 the one upper), a pump mounted between the tanks can pump the water from i-th reservoir to the storage and from the storage to the $i+1$-th reservoir in any given mode.

Modes of operation of pumps - the pump, pumping water through the upper inlet of the first tank; the pump, pumping water through the tank drives; pumps, pumping water into the tanks through their lower inputs, as well as the pump to pump water from the last tank – will consider on the assumption that the conditions for every tank i:

1. Water Movement in a reservoir is described by the equation (9) and formulas (7), (8);

2. At moment of time t_j tank i fills to the level ${}^{ij}h$. It's enough to, before the moment t_j at point x_i (at the left wall) the following condition are reasonable for a long time $Q^{ij-0} + q^{ij-0}\Delta x_i = \left({}^{ij}h\right)^{5/3} \dfrac{w_i\sqrt{S_{0i}}}{n_i}$, where $Q^{ij-0} + q^{ij-0}\Delta x_i$ – the constant intensity of inflow (consumption) water in the tank i until the

moment t_j, and at $x_i + \Delta x_i$ (at right wall) it is provided by the free flow condition (water freely flows into the storage);

3. Since time $t_j + 0$ and up to the moment $t_j + \Delta t_j$ into the tank i water is pumping. In this case pumps work in such a way that during the time Δt_j , in point x_i water depth is constant and equal to h^{ij}. Value h^{ij} can be found from the equation $Q^{ij} + q^{ij}\Delta x_i = \left(h^{ij}\right)^{5/3} \frac{w_i\sqrt{S_{0i}}}{n_i}$, where $Q^{ij} + q^{ij}\Delta x_i$ – the constant of the intensity of the inflow into the tank i from moment $t_j + 0$ to moment $t_j + \Delta t_j$. In $x_i + \Delta x_i$ all this time is provided by the free flow condition, which corresponds to the outflow of water in the tank with the intensity $Q^{i+1-0j}(t) = h\left(x_i + \Delta x_i, t\right)^{5/3} \frac{w_i\sqrt{S_{0i}}}{n_i}$.

4. After finding the function $h(x,t)$ for $\left\{x_i \le x \le x_i + \Delta x_i;\, t_j < t \le t_j + \Delta t_j\right\}$, we can found $Q^{i+1j} = \frac{1}{\Delta t_j}\int_{t_j}^{t_j+\Delta t_j} Q^{i+1-0j}(t)dt = \frac{1}{\Delta t_j}\frac{w_i\sqrt{S_{0i}}}{n_i}\int_{t_j}^{t_j+\Delta t_j} h\left(x_i + \Delta x_i, t\right)^{5/3} dt$ – average by the time amount of the flow near the right wall of the tank i. With this constant flow of water through the upper input, we will look for a solution for the tank $i+1$. By adding the constant $q^{i+1j}\Delta x_{i+1} = \frac{1}{\Delta t_j}\int_{x_{i+1}}^{x_{i+2}}\int_{t_j}^{t_{j+1}} q(x,t)dtdx$ (flow from an independent source) to Q^{i+1j}–, we obtain the total magnitude of the influx water in the tank $i+1$ on the interval from $t_j + 0$ to $t_j + \Delta t_j$. Repeat steps 3, 4 for the tank $i+1$. Similarly on the time interval from $t_j + 0$ to $t_j + \Delta t_j$ the task can be solved for all tanks. If we know function $h(x,t)$, we can find ${}^{ij+1}h = \frac{1}{\Delta x_i}\int_{x_i}^{x_i+\Delta x_i} h\left(x, t_{j+1}\right)dx$ – its average amount within the limits for every tank i at the moment t_{j+1}, corresponding to steady state, similar to described above. We came to the situation for t_{j+1}, a similar for the situation at the moment t_j … By such repeating the paragraphs. 3-5, we can determine the hydraulic condition of the river at any arbitrarily large time.

Numerical scheme

D - is rectangular area of the independent variables x and t, consider that the counting grid is uniform on x, and on t. Each cell of the grid is identifying indices of her left lower node: i – index on x and j – index on t; the number of intervals on x is assumed to be I, number of intervals on t is equally

J; position of the node in the grid in D will be specified of its spatial coordinate x_i, $i \in \{0,1,...I\}$, and the same is for time coordinate t_j, $j \in \{0,1,...J\}$.

Within the limits of cell (i,j), $i \in \{0,1,...I-1\}$, $j \in \{0,1,...J-1\}$, consider to call the bottom edge is the edge with the index i, the upper edge – the edge with the index $i+1-0$, left edge – the edge with the index j, right edge – the edge with the index $j+1-0$. The length of the horizontal edges of the cell is equal to $\Delta x_i = x_{i+1} - x_i$ and does not depend on i, length of the vertical edges is equal to $\Delta t_j = t_{j+1} - t_j$ and does not depend on j.

Consider that the original data $w(x,t)$, $n(x,t)$, $S_0(x,t)$ are positive and constant functions inside each cell (i,j), piecewise constant function in the area D. Values w, n и S_0 inside the cell (i,j) consider to denote as w_i, n_i and S_{0i}. Consider that external source $q(x,t)$ is centered on the vertical edges of the cells, so that,

$q(x,t) = \begin{cases} 0, \ x \neq x_i \\ q^{ij}(t), \ x = x_i \end{cases}$, where $q^{ij}(t)$ – piecewise constant function that takes constant values q^{ij} on the interval $t_j < t < t_{j+1}$.

On each cell consider the depth $h(x,t)$ on the vertical edge j is the known constant $h^{ij} > 0$, and on the horizontal edge i is the known constant ${}^{ij}h > 0$. In the situation $h^{ij} = {}^{ij}h$ inside the cell and on the edges $i+1-0$, $j+1-0$ is performed $h = h^{ij} = {}^{ij}h$. But in the situation $h^{ij} \neq {}^{ij}h$ three cases are possible (see. Fig. 2):

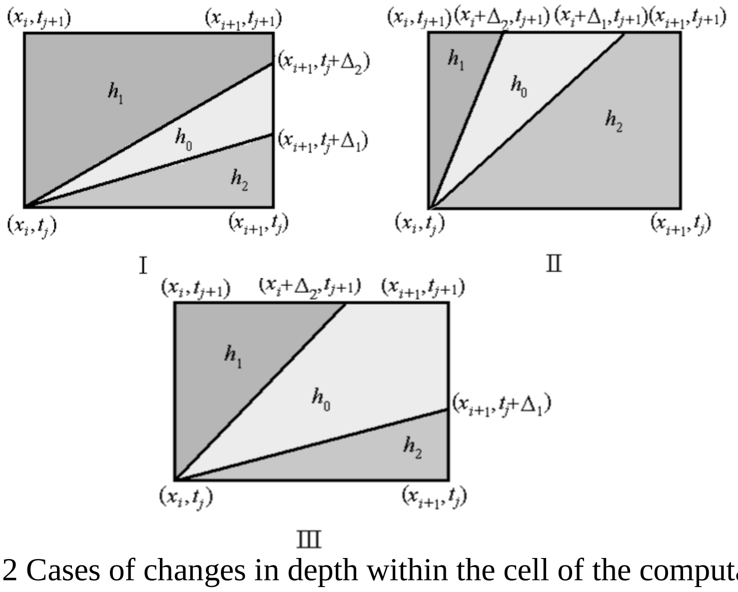

Fig. 2 Cases of changes in depth within the cell of the computational grid

Results

Using described above method we have solved several test tasks from [4] and compare the results with analytical solutions. Results on the Fig.3 and Fig.4 shows the function $h(x,t)$ distribution for each of 10 cell in comparing with pictures given in [4].

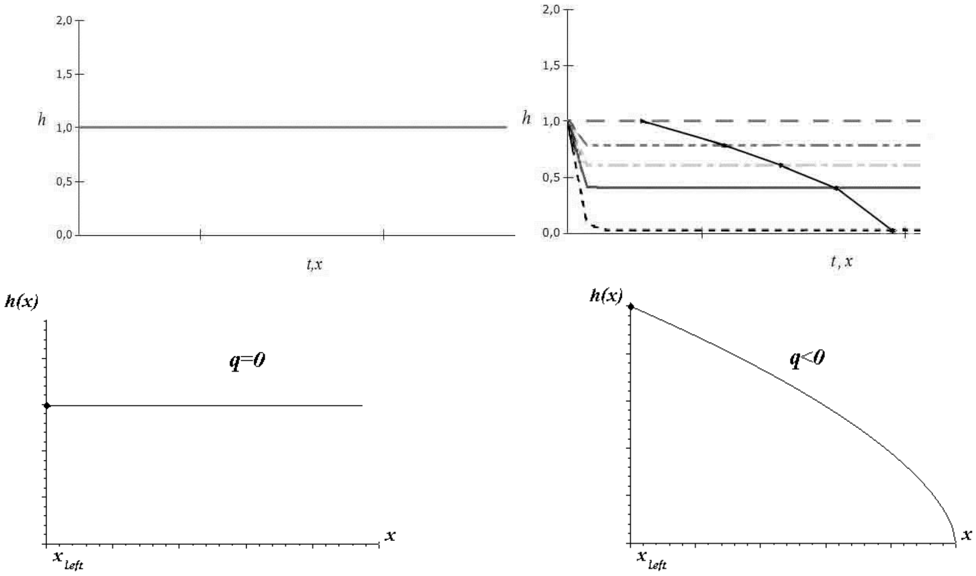

Fig. 3. $h(x)$ depending on the sign of q for constants w, n, S_0, q

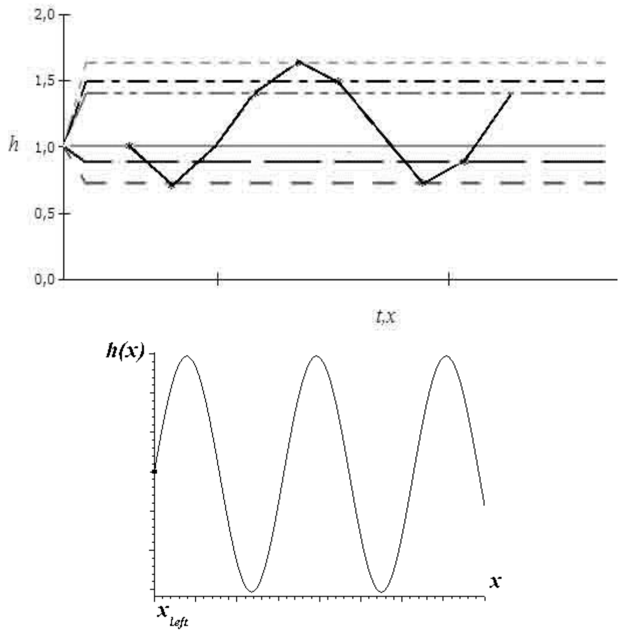

Fig. 4. $h(x)$ with constant w, S_0, $q = 0$ and $n(x) = a + b\sin(x)$

Here we have good correspondence in results. Thus, we can consider that such numerical scheme could be used for more complex computations.

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